Introduction to Dempster-Shafer Theory of Belief Functions

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- DS encompasses probability theory and set-membership approaches as special cases.
- It is very general: many applications in AI (expert systems, machine learning), engineering (information fusion, uncertainty quantification, risk analysis), statistical inferences, etc.
- Evidential reasoning can be applied to very large problems.

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- **Uncertainty**

  Example: "I think John is 1.8m tall"
  In this case, the piece of information John is 1.8m tall is precise but uncertain

- **Imprecision**

  Example: "John is between 1.7m and 1.9m tall"
  In this case, the piece of information "John is between 1.7m and 1.9m tall" is certain but imprecise

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- **Imprecision and uncertainty**

  Example: "I think John is between 1.7m and 1.9m tall"
  In this case, the piece of information "John is between 1.7m and 1.9m tall" is both uncertain and imprecise

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- **Uncertainty** $\Rightarrow$ Classicaly tackled with probabilities

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- **Uncertainty** ⇒ Classicaly tackled with probabilities

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- **Imprecision** ⇒ Classicaly tackled with sets

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- **Imprecision and uncertainty**

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Aleatory uncertainty (Randomness)

vs

Epistemic uncertainty (Lack of knowledge)

Introduction

Difficulties to represent ignorance with probabilities

"Le tiercé c'est mon dada" (O. Sharif)

- Consider a **horse race** with three horses h_1, h_2 and h_3

Expert 1: "All three horses have an equal chance of winning (same level)"

Model: $p(\{h_1\}) = p(\{h_2\}) = p(\{h_3\}) = \frac{1}{3}$

Expert 2: "I have no idea (complete ignorance)"

Model: $p(\{h_1\}) = p(\{h_2\}) = p(\{h_3\}) = \frac{1}{3}$

Le Derby d'Epsom (Géricault)

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Le Derby d'Epsom (Géricault)

- Problem: Two distinct pieces of information are modeled identically.
- There is a need for a richer model.

Outline

Outline

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- Let us consider a variable of interest X taking its values into a finite set of hypotheses $\Omega = \{\omega_1, ..., \omega_K\}$ called the universe or the frame of discernment.
  - Example: the horse that will win the race. $\Omega = \{h_1, h_2, h_3\}$
- A piece of information regarding the value ω_0 taken by this variable can be represented using a mass function (MF) m defined as a mapping $m : 2^\Omega \to [0, 1]$ verifying

$$\sum_{A \subseteq \Omega} m^\Omega(A) = 1 \ .$$

- The real $m(A)$ represents the part of belief allocated to the hypothesis that the searched true value ω_0 belongs to A and nothing more.
- A set A s.t. $m(A) > 0$ is called a focal set of m.

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"Si vous avez perdu au tiercé, vengez-vous. Mangez du cheval." (P. Dac)

- Let us consider again the **horse race** example with $\Omega = \{h_1, h_2, h_3\}$

Expert 1: "All three horses have an equal chance of winning (same level)"

Model: $m(\{h_1\}) = m(\{h_2\}) = m(\{h_3\}) = \frac{1}{3}$

Expert 2: "I have no idea (complete ignorance)"

Model: $m(\{h_1, h_2, h_3\}) = 1$

Le Derby d'Epsom (Géricault)

- If the evidence tells us that the truth is in $A \subseteq \Omega$ for sure, then we have a logical or categorical mass function m_A s.t. $m_A(A) = 1$.
- m_Ω represents the total ignorance, it is called the vacuous mass function
- If all focal sets of m are singletons, m is said to be Bayesian. It is equivalent to a probability distribution.
- A mass function can thus be seen as:
  - a generalized set
  - a generalized probability distribution

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- A MF m is in one-to-one correspondence (each function represents the same information) with :
  - a belief function Bel defined for all $A \subseteq \Omega$ by:

$$Bel(A) = \sum_{\emptyset \neq B \subseteq A} m(B),$$

    $Bel(A)$ represents the total degree of belief supporting the fact that $\omega_0 \in A$ (Total support in A)

  - a plausibility function Pl defined for all $A \subseteq \Omega$ by:

$$Pl(A) = \sum_{A \cap B \neq \emptyset} m(B) = Bel(\Omega) - Bel(\overline{A})$$

    with $\overline{A} = \Omega \setminus A$.
    $Pl(A)$ represents the total sum of beliefs that are not in contradiction with A (Consistency with A)

Representation of information

Example

With $\Omega = \{a, b, c\}$, $m(\{a\}) = 0.3$, $m(\{b\}) = 0.4$ and $m(\Omega) = 0.3$

Let us compute $Bel(\{a, b\})$ as an example, we have

$$Bel(\{a, b\}) = \sum_{B:\emptyset \neq B \subseteq \{a,b\}} m(B)$$
$$= m(\{a\}) + m(\{b\}) + m(\{a, b\}) = .7$$

For Pl, let us compute $Pl(\{a, b\})$ as an example as well

$$Pl(\{a, b\}) = \sum_{B:B \cap \{a,b\} \neq \emptyset}$$
$$= m(\{a\}) + m(\{b\}) + m(\{a, b\}) + m(\{a, c\}) + m(\{b, c\}) + m(\Omega)$$
$$= 1$$

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With $\Omega = \{a, b, c\}$, $m(\{a\}) = 0.3$, $m(\{b\}) = 0.4$ and $m(\Omega) = 0.3$

bin.order		*m*	*Bel*	*Pl*
000	$\emptyset$			
001	a	.3		
010	b	.4		
011	a, b			
100	c			
101	a, c			
110	b, c			
111	a, b, c	.3		

```
> library(ibelief)

> m=c(0,.3,.4,0,0,0,0,.3)

> pl=mtopl(m)

> bel=mtobel(m)
```


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Example: With $\Omega = \{a, b, c\}$, $m(\{a\}) = 0.3$, $m(\{b\}) = 0.4$ and $m(\Omega) = 0.3$

bin.order		*m*	*Bel*	*Pl*
000	$\emptyset$			
001	a	.3	.3	.6
010	b	.4	.4	.7
011	a, b		.7	1
100	c			.3
101	a, c		.3	.6
110	b, c		.4	.7
111	a, b, c	.3	1	1

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- $Bel(\emptyset) = Pl(\emptyset) = 0$
- $Bel(\Omega) \leq 1$ and $Pl(\Omega) \leq 1$ (as $m(\emptyset)$ could be positive)
- $Bel(A) \leq Pl(A)$
- $Pl(A) = 1 - Bel(\overline{A})$
- If m is Bayesian (i.e. all focal elements are singletons) then $Bel = Pl$ is a probability measure

- The uncertainty about a proposition A is represented by two numbers: $Bel(A)$ and $Pl(A)$, with $Bel(A) \leq Pl(A)$.
- The intervals $[Bel(A), Pl(A)]$ have maximum length when m is vacuous ($m = m_\Omega$).
  - In this case: $Bel(A) = 0$ for all $A \neq \Omega$ and $Pl(A) = 1$ for all $A \neq \emptyset$
- The intervals $[Bel(A), Pl(A)]$ have minimum length when m is Bayesian.
  - In this case, for all A:

$$Bel(A) = Pl(A) = \sum_{\omega \in A} m(\omega)$$

and Bel and Pl are probability measures.

- If m has its focal elements nested ($A_1 \subset A_2 \subset \ldots \subset A_n$, with A_i, $i \in \{1, 2, \ldots, n\}$ the focal elements of m), m is said to be consonant.
- In this case, for all $A \subseteq \Omega$, $B \subseteq \Omega$:

$$Bel(A \cap B) = min(Bel(A), Bel(B))$$

and

$$Pl(A \cup B) = max(Pl(A), Pl(B))$$

meaning Pl is a possibility measure and Bel is its dual necessity measure.

Outline

Representation of information

Combining information

Decision making

Conclusion

Cunjunctive Rule of Combination

- Two mass functions m_1 and m_2 from two reliable and distinct sources of information can be combined using the conjunctive rule of combination (CRC) defined by:

$$(m_1 \circledcap m_2)(A) = m_{1 \circledcap 2}(A) = \sum_{B \cap C = A} m_1(B) \cdot m_2(C), \quad \forall A \subseteq \Omega \,.$$

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Cunjunctive Rule of Combination

Example

With $\Omega = \{a, b, c\}$, let us consider a MF m_1 and another independent MF m_2 s.t.

$$\begin{cases} m_1(\{b\}) & = & .4 \\ m_1(\{a,b\}) & = & .6 \end{cases} \text{ and } \begin{cases} m_2(\{b,c\}) & = & .3 \\ m_2(\{a,b,c\}) & = & .7 \end{cases}$$

CRC	$m_2(\{b,c\}) = .3$	$m_2(\{a,b,c\}) = .7$
$m_1(\{b\}) = .4$	$\{b\} \cap \{b,c\} = \{b\}$ $.4 \times .3 = .12$	$\{b\} \cap \{a,b,c\} = \{b\}$ $.4 \times .7 = .28$
$m_1(\{a,b\}) = .6$	$\{a,b\} \circledcap \{b,c\} = \{b\}$ $.6 \times .3 = .18$	$\{a,b\} \circledcap \{a,b,c\} = \{a,b\}$ $.6 \times .7 = .42$

The CRC $m = m_1 \circledcap m_2$ is given by

- $m(\{b\}) = .4 \times .3 + .4 \times .7 + .6 \times .3 = .58$
- $m(\{a,b\}) = .6 \times .7 = .42$

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Dempster's rule

- **If and only if m_1 and m_2 are two reliable and distinct mass functions** (Axiomatic justifications, see e.g. Smets 2007)
- Dempster's rule := CRC normalized: $m_{1\oplus 2}(\emptyset) = 0$ and

$$(m_1 \oplus m_2)(A) = m_{1\oplus 2}(A) = \frac{1}{1-\kappa} \sum_{B \cap C = A} m_1(B) \cdot m_2(C), \quad \forall A \neq \emptyset$$

with $\kappa = \sum_{B \cap C = \emptyset} m_1(B) \cdot m_2(C)$ (called degree of conflict).

Disjunctive Rule of Combination

- If the sources are distinct but only one of the sources is reliable (and we don't know which one), the disjunctive rule of combination (DRC) defined as follows can be applied:

$$(m_1 \ocircle\!\!\!\!\cup\, m_2)(A) = m_{1 \ocircle\!\!\!\!\cup 2}(A) = \sum_{B \cup C = A} m_1(B) \cdot m_2(C), \quad \forall A \subseteq \Omega \,.$$

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Properties for these rules

- With these rules $\circledcap$, $\oplus$ and $\circledcup$, the order the sources are combined does not change the results
- $m_1 \circledcap m_2 = m_2 \circledcap m_1$ (Commutativity) (Likewise for $\oplus$ and $\circledcup$)
- $(m_1 \circledcap m_2) \circledcap m_3 = m_1 \circledcap (m_2 \circledcap m_3)$ (Associativity) (Likewise for $\oplus$ and $\circledcup$)

Combining information
Example

	m_1	m_2	$m_1 \circledcap m_2$	$m_1 \oplus m_2$	$m_1 \circledcup m_2$
$\emptyset$					
a	.3				
b	.4				
a, b		.5			
c					
a, c		.1			
b, c					
a, b, c	.3	.4			

```
> m1=c(0,.3,.4,0,0,0,0,.3)

> m2=c(0,0,0,.5,0,.1,0,.4)

> mcunjunctive = DST(cbind(m1,m2),1)

> mdempster = DST(cbind(m1,m2),2)

> mdisjunctive = DST(cbind(m1,m2),4)
```

Combining information

Example

	m_1	m_2	$m_1 \circledcap m_2$	$m_1 \oplus m_2$	$m_1 \circledcup m_2$
$\emptyset$			.04		
a	.3		.30	.312	
b	.4		.36	.375	
a, b		.5	.15	.156	.35
c					
a, c		.1	.03	.031	.03
b, c					
a, b, c	.3	.4	.12	.125	.62

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Misconception about Dempster's rule

- Following an old report from Zadeh (1979) - it is still nowadays repeated that "Dempster's rule yields counterintuitive results" (usually used as a justification to introduce new combination rules)
- Zadeh's example: $\Omega = \{a, b, c\}$, two experts reporting:
  - Expert 1: $m_1(\{a\}) = 0.99$, $m_1(\{b\}) = 0.01$ and $m_1(\{c\}) = 0$
  - Expert 2: $m_2(\{a\}) = 0$, $m_2(\{b\}) = 0.01$ and $m_2(\{c\}) = 0.99$
- Then $m_{1 \textcircled{\cup} 2}(b) = 1$, which is claimed to be "counterintuitive" by some authors because both experts considered b as very unlikely.
- But:
  - Both experts are totally reliable.
  - Expert 1 indicates that c is absolutely impossible.
  - Expert 2 indicates that a is absolutely impossible.
- Then b is the only possibility. We are in a situation, which is possible for both experts, where the true answer is b.
- Dempster's rule does produce sound results when used cordance with the axioms, from which it derived.

Discounting

A simple correction example (Shafer, 1976).

Discounting of a mass function (MF) m is defined by (Shafer,1976):

$$\begin{cases} {}^{\alpha}m(A) & = & (1-\alpha)m(A), \quad \forall A \subset \Omega, \\ {}^{\alpha}m(\Omega) & = & (1-\alpha)m(\Omega) + \alpha \ , \end{cases}$$

where $\alpha \in [0,1]$ is the discount rate.

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where $\alpha \in [0,1]$ is the discount rate.

Example:

- $\Omega = \{a, b, c\}$
- $m(\{a\}) = .2$, $m(\{b\}) = .4$ and $m(\{a,b\}) = .4$
- With discount rate $\alpha = .2$:

$$\begin{cases} {}^{\alpha}m(\{a\}) &= .8 \times .2 &= .16 \\ {}^{\alpha}m(\{b\}) &= .8 \times .4 &= .32 \\ {}^{\alpha}m(\{a,b\}) &= .8 \times .4 &= .32 \\ {}^{\alpha}m(\Omega) &= .8 \times .0 + .2 &= .20 \end{cases}$$

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```
> m=c(0,.2,.4,.4,0,0,0,0)

> mdisc = discounting(m,.8)
```

L'argument placé dans cette fonction est $1 - \alpha = .8$ qui est le degré de fiabilité de la source (80% des masses sont gardées dans ce cas)

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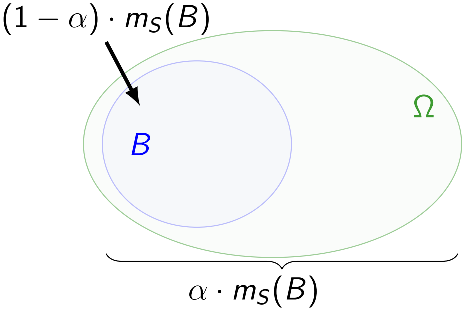
For each focal element B of m_S:



- A part $(1-\alpha) \cdot m_S(B)$ remains on B.
- A part $\alpha \cdot m_S(B)$ is transferred to Ω.

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Discounting

Matrix representation (Smets, 2002)

Discounting ${}^{\alpha}m$ is a generalization of m (${}^{\alpha}m \sqsupseteq_s m$):

$$ {}^{\alpha}m(A) = \sum_{B \subseteq \Omega} {}^{\alpha}G(A,B)m(B) \ , $$

with ${}^{\alpha}\mathbf{G}$ a generalisation matrix defined by:

$$ {}^{\alpha}G(A,B) = \begin{cases} 1-\alpha & \text{if } A = B \neq \Omega, \\ \alpha & \text{if } A = \Omega \text{ and } B \subset A, \\ 1 & \text{if } A = B = \Omega \\ 0 & \text{otherwise.} \end{cases} $$

$$ {}^{\alpha}\mathbf{G} = \begin{pmatrix} 1-\alpha & 0 & \dots & 0 & 0 \\ 0 & 1-\alpha & \dots & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & \dots & 1-\alpha & 0 \\ \alpha & \alpha & \dots & \alpha & 1 \end{pmatrix} $$

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Discounting

Matrix representation: example

With $\alpha = .2$, $\beta = 1 - \alpha = .8$ and $\Omega = \{a, b, c\}$:

$$
\begin{array}{ll}
 & {}^{\alpha}m \quad = \quad {}^{\alpha}G(A,B) \quad \cdot \quad m \\
\begin{array}{l}
000 : \emptyset \\
001 : \{a\} \\
010 : \{b\} \\
011 : \{a, b\} \\
100 : \{c\} \\
101 : \{a, c\} \\
110 : \{b, c\} \\
111 : \{a, b, c\}
\end{array}
&
\begin{pmatrix} .0 \\ .16 \\ .32 \\ .32 \\ .0 \\ .0 \\ .0 \\ .20 \end{pmatrix}
=
\begin{pmatrix}
\beta & & & & & & & \\
 & \beta & & & & & & \\
 & & \beta & & & & & \\
 & & & \beta & & & & \\
 & & & & \beta & & & \\
 & & & & & \beta & & \\
 & & & & & & \beta & \\
\alpha & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha & 1
\end{pmatrix}
\cdot
\begin{pmatrix} .0 \\ .2 \\ .4 \\ .4 \\ .0 \\ .0 \\ .0 \\ .0 \end{pmatrix}
\end{array}
$$

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Outline

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- A way to make a **hard decision** is to choose a decision $d = \omega \in \Omega$ maximizing a probability transform of m, as for example using the Pignistic transform $BetP$ defined by:

$$BetP(\{\omega\}) = \sum_{A \subseteq \Omega, \omega \in A} \frac{m(A)}{|A|\,(1 - m(\emptyset))}, \quad \forall \omega \in \Omega\ .$$

- Example: With $\Omega = \{a, b, c\}$, $m(\{a\}) = 0.3$, $m(\{b\}) = 0.4$ and $m(\Omega) = 0.3$

$$BetP = \begin{cases} \{a\} \mapsto 0.3 + \frac{0.3}{3} = 0.4 \\ \{b\} \mapsto 0.4 + \frac{0.3}{3} = 0.5 \\ \{c\} \mapsto 0.0 + \frac{0.3}{3} = 0.1 \end{cases}$$

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```
> m=c(0,.3,.4,0,0,0,0,.3)
> betp = mtobetp(m)
```


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- A way to make a **partial decision** is to choose a set-valued decision $d = A \subseteq \Omega$ composed of elements of Ω, which are **not dominated according to a preference relation**:

1. The relation of **strong dominance or interval dominance** defined by

$$\omega \succeq_{sd} \omega' \iff Bel(\{\omega\}) \geq Pl(\{\omega'\})$$

2. The relation of **weak dominance** defined by

$$\omega \succeq_{wd} \omega' \iff Bel(\{\omega\}) \geq Bel(\{\omega'\}) \; and \; Pl(\{\omega\}) \geq Pl(\{\omega'\})$$

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Decision making

An example with partial decisions using the strong and weak dominance criteria

Example: With $\Omega = \{a, b, c\}$, $m(\{a\}) = 0.3$, $m(\{b\}) = 0.4$ and $m(\Omega) = 0.3$

$$Bel = \begin{cases} \{a\} \mapsto 0.3 \\ \{b\} \mapsto 0.4 \\ \{c\} \mapsto 0.0 \end{cases} \qquad Pl = \begin{cases} \{a\} \mapsto 0.6 \\ \{b\} \mapsto 0.7 \\ \{c\} \mapsto 0.3 \end{cases}$$

Relation SD	Non-dominated
$Bel(\{b\}) = .4 \not\geq Pl(\{a\}) = .6$ and $Bel(\{c\}) = 0 \not\geq Pl(\{a\}) = .6$	a
$Bel(\{a\}) = .3 \not\geq Pl(\{b\}) = .7$ and $Bel(\{c\}) = 0 \not\geq Pl(\{b\}) = .7$	b
$Bel(\{a\}) = .3 \geq Pl(\{c\}) = .3$ (so $a \succeq_{sd} c$)	-

Conclusion using SD: $d = \{a, b\}$.

Relation WD	Dominated
$Bel(\{b\}) = 0.4 \geq Bel(\{a\}) = 0.3$ and $Pl(\{b\}) = 0.7 \geq Pl(\{a\}) = 0.6$ (so $b \succeq_{wd} a$)	a
$Bel(\{b\}) = 0.4 \geq Bel(\{c\}) = 0.0$ and $Pl(\{b\}) = 0.7 \geq Pl(\{c\}) = 0.3$ (so $b \succeq_{wd} c$)	c

Conclusion using WD: $d = \{b\}$.

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Outline

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Conclusion

- Dempster-Shafer (DS) theory of belief functions is a flexible mathematical framework for dealing with imperfect information.
- It encompasses probability theory and set-membership approaches as special cases.
- Belief functions can be seen as weighted opinions.

References to start learning Belief functions

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Thank you for your attention.