Belief function correction mechanisms

David Mercier, Thierry Denœux, and Marie-Hélène Masson

Abstract Different operations can be used in the theory of belief functions to correct the information provided by a source, given metaknowledge about that source. Examples of such operations are discounting, de-discounting, extended discounting and contextual discounting. In this article, the links between these operations are explored. New interpretations of these schemes, as well as two families of belief function correction mechanisms are introduced and justified. The first family generalizes previous non-contextual discounting operations, whereas the second generalizes the contextual discounting.

Key words: Dempster-Shafer theory, Belief functions, discounting operations, disjunctive and conjunctive canonical decompositions.

1 Introduction

Introduced by Dempster [1] and Shafer [13], belief functions constitute one of the main frameworks for reasoning with imperfect information.

When receiving a piece of information represented by a belief function, some metaknowledge regarding the quality or reliability of the source that provides the

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information, can be available. To correct the information according to this meta-knowledge, different tools can be used:

- The *discounting operation*, introduced by Shafer in his seminal book [13], allows one to weaken the information provided by the source;
- The *de-discounting operation*, introduced by Denœux and Smets [2], has the effect of strengthening the information;
- The *extended discounting operation*, introduced by Zhu and Basir [20], makes it possible to weaken, strengthen or contradict the information;
- The *contextual discounting operation*, a refining of the discounting operation, introduced by Mercier et al. [10], weakens the information by taking into account more detailed knowledge regarding the reliability of the source in different contexts, i.e., conditionally on different hypotheses regarding the answer to the question of interest.

In this article, the links between these operations are explored. Belief function correction mechanisms encompassing these schemes are introduced and justified.

First, discounting, de-discounting, and extended discounting are shown to be particular cases of a parameterized family of transformations [9]. This family includes all possible transformations, expressed by a belief function, based on the different states in which the source can be when the information is supplied.

Secondly, another family of correction mechanisms based on the concepts of negation [4] and canonical decompositions [16, 3, 12] of a belief function is explored. This family is shown to generalize the contextual discounting operation.

Belief functions are used in different theories of uncertainty such as, for instance, models based on lower and upper probabilities including Dempster's model [1] and the related Hint model [7], random set theory [6], or the Transferable Belief Model developed by Smets [15, 19]. In the latter model, belief functions are interpreted as weighted opinions of an agent or a sensor. This model is adopted in this article.

This article is organized as follows. Background material on belief functions is recalled in Section 2. All the discounting operations are presented in Section 3. A new interpretation of non-contextual discounting as well as a parameterized family of correction mechanisms are introduced and justified in Section 4. Another family of correction mechanisms based on the disjunctive and conjunctive canonical decompositions of a belief function is presented in Section 5. In Section 6, an example of a correction mechanism introduced in this article is tested with real data in a postal address recognition application, in which decisions associated with confidence scores are combined. Finally, Section 7 concludes this paper.

2 Belief functions: basic concepts

2.1 Representing information

Let us consider an agent Ag in charge of making a decision regarding the answer to a given question Q of interest.

Let $\Omega = \{\omega_1, \ldots, \omega_K\}$, called the *frame of discernment*, be the finite set containing the possible answers to question Q.

The information held by agent Ag regarding the answer to question Q can be quantified by a *basic belief assignment (BBA) or mass function* m^{Ω}_{Ag}, defined as a function from 2^{Ω} to $[0,1]$, and verifying:

$$\sum_{A \subseteq \Omega} m^{\Omega}_{Ag}(A) = 1 \ . \tag{1}$$

Function m^{Ω}_{Ag} describes the state of knowledge of agent Ag regarding the answer to question Q belonging to Ω. By extension, it also represents an item of evidence that induces such a state of knowledge. The quantity $m^{\Omega}_{Ag}(A)$ is interpreted as the part of the unit mass allocated to the hypothesis: "the answer to question Q is in the subset A of Ω".

When there is no ambiguity, the full notation m^{Ω}_{Ag} will be simplified to m^{Ω}, or even m.

Definition 1. The following definitions are considered.

- A subset A of Ω such that $m(A) > 0$ is called a *focal element* of m.
- A BBA m with only one focal element A is said to be *categorical* and is denoted m_A; we thus have $m_A(A) = 1$.
- Total ignorance is represented by the BBA m_{Ω}, called the *vacuous belief function*.
- A BBA m is said to be:
  - *dogmatic* if $m(\Omega) = 0$;
  - *non-dogmatic* if $m(\Omega) > 0$;
  - *normal* if $m(\emptyset) = 0$;
  - *subnormal* if $m(\emptyset) > 0$;
  - *simple* if m has no more than two focal sets, Ω being included.

Finally, $\overline{m}$ denotes the *negation* of m [4], defined by $\overline{m}(A) = m(\overline{A})$, for all $A \subseteq \Omega$.

2.2 Combining pieces of information

Two BBAs m_1 and m_2 induced by distinct and reliable sources of information can be combined using the *conjunctive rule of combination (CRC)*, also referred to as the *unnormalized Dempster's rule of combination*, defined for all $A \subseteq \Omega$ by:

$$m_1 \textcircled{\cap} m_2(A) = \sum_{B\cap C=A} m_1(B)m_2(C) \ . \tag{2}$$

Alternatively, if we only know that at least one of the sources is reliable, BBAs m_1 and m_2 can be combined using the *disjunctive rule of combination (DRC)*, defined for all $A \subseteq \Omega$ by:

$$m_1 \textcircled{\cup} m_2(A) = \sum_{B\cup C=A} m_1(B)m_2(C) \ . \tag{3}$$

2.3 Marginalization and vacuous extension

A mass function defined on a product space $\Omega \times \Theta$ may be *marginalized* on Ω by transferring each mass $m^{\Omega\times\Theta}(B)$ for $B \subseteq \Omega \times \Theta$ to its projection on Ω:

$$m^{\Omega\times\Theta\downarrow\Omega}(A) = \sum_{\substack{B\subseteq\Omega\times\Theta,\\ \mathrm{Proj}(B\downarrow\Omega)=A}} m^{\Omega\times\Theta}(B), \tag{4}$$

for all $A \subseteq \Omega$ where $\mathrm{Proj}(B \downarrow \Omega)$ denotes the projection of B onto Ω.

Conversely, it is usually not possible to retrieve the original BBA $m^{\Omega\times\Theta}$ from its marginal $m^{\Omega\times\Theta\downarrow\Omega}$ on Ω. However, the *least committed*, or *least informative BBA* [14] such that its projection on Ω is $m^{\Omega\times\Theta\downarrow\Omega}$ may be computed. This defines the *vacuous extension* of m^{Ω} in the product space $\Omega \times \Theta$ [14], noted $m^{\Omega\uparrow\Omega\times\Theta}$, and given by:

$$m^{\Omega\uparrow\Omega\times\Theta}(B) = \begin{cases} m^{\Omega}(A) & \text{if } B = A\times\Theta, A \subseteq \Omega, \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

2.4 Conditioning and ballooning extension

Conditional beliefs represent knowledge that is valid provided that an hypothesis is satisfied. Let m be a mass function and $B \subseteq \Omega$ an hypothesis; the *conditional belief function* $m[B]$ is given by:

$$m[B] = m \textcircled{\cap} m_B. \tag{6}$$

If $m^{\Omega\times\Theta}$ is defined on the product space $\Omega \times \Theta$, and θ is a subset of Θ, the conditional BBA $m^{\Omega}[\theta]$ is defined by combining $m^{\Omega\times\Theta}$ with $m_{\theta}^{\Theta\uparrow\Omega\times\Theta}$, and marginalizing the result on Ω:

$$m^{\Omega}[\theta] = \left(m^{\Omega\times\Theta} \textcircled{\cap} m_{\theta}^{\Theta\uparrow\Omega\times\Theta} \right)^{\downarrow\Omega} . \tag{7}$$

Assume now that $m^{\Omega}[\theta]$ represents the agent's beliefs on Ω conditionally on θ, i.e., in a context where θ holds. There are usually many BBAs on $\Omega \times \Theta$, whose conditioning on θ yields $m^{\Omega}[\theta]$. Among these, the least committed one is defined for all $A \subseteq \Omega$ by:

$$m^{\Omega}[\theta]^{\Uparrow\Omega\times\Theta}(A\times\theta\cup\Omega\times\overline{\theta}) = m^{\Omega}[\theta](A). \quad (8)$$

This operation is referred to as the *deconditioning* or *ballooning extension* [14] of $m^{\Omega}[\theta]$ on $\Omega\times\Theta$.

3 Correction mechanisms

3.1 Discounting

When receiving a piece of information represented by a mass function m, agent Ag may have some doubts regarding the reliability of the source that provided this information. Such metaknowledge can be taken into account using the discounting operation introduced by Shafer [13, page 252], and defined by:

$${}^{\alpha}m = (1-\alpha)m + \alpha\, m_{\Omega} \;, \quad (9)$$

where $\alpha \in [0,1]$.

A discount rate α equal to 1, means that the source is not reliable and the piece of information it provides cannot be taken into account, so Ag's knowledge remains vacuous: $m^{\Omega}_{Ag} = {}^{1}m = m_{\Omega}$. On the contrary, a null discount rate indicates that the source is fully reliable and the piece of information is entirely accepted: $m^{\Omega}_{Ag} = {}^{0}m = m$. In practice, however, agent Ag usually does not know for sure whether the source is reliable or not, but has some degree of belief expressed by:

$$\begin{cases} m^{\mathscr{R}}_{Ag}(\{R\}) = 1-\alpha \\ m^{\mathscr{R}}_{Ag}(\mathscr{R}) \quad = \alpha, \end{cases} \quad (10)$$

where $\mathscr{R} = \{R, NR\}$, R and NR standing, respectively, for *"the source is reliable"* and *"the source is not reliable"*. This formalization yields expression (9), as demonstrated by Smets in [14, Section 5.7].

Let us consider a BBA m^{Ω}_{0} defined by

$$m^{\Omega}_{0}(A) = \begin{cases} \beta & \text{if } A = \emptyset \\ \alpha & \text{if } A = \Omega \\ 0 & \text{otherwise,} \end{cases} \quad (11)$$

with $\alpha \in [0,1]$ and $\beta = 1-\alpha$. The discounting operation (9) of a BBA m is equivalent to the disjunctive combination (3) of m with m^{Ω}_{0}. Indeed:

$$m \bigcirc\!\!\!\!\cup\, m^{\Omega}_{0}(A) = m(A)m^{\Omega}_{0}(\emptyset) = \beta m(A) = {}^{\alpha}m(A),\; \forall A \subset \Omega \;,$$

and

$$m \bigcirc\!\!\!\!\cup\, m_0^\Omega(\Omega) = m(\Omega) m_0^\Omega(\emptyset) + m_0^\Omega(\Omega) \sum_{A \subseteq \Omega} m(A) = \beta m(\Omega) + \alpha = {}^\alpha m(\Omega).$$

3.2 De-Discounting

In this process, agent *Ag* receives a piece of information ${}^\alpha m$ from a source S, different from m_Ω and discounted with a discount rate $\alpha < 1$.

If *Ag* knows the discount rate α, then it can recompute m by reversing the discounting operation (9):

$$m_{Ag} = m = \frac{{}^\alpha m - \alpha\, m_\Omega}{1-\alpha}\ . \tag{12}$$

This procedure is called *de-discounting* by Denœux and Smets in [2].

If the agent receives a mass function m discounted with an unknown discount rate α, it can imagine all possible values in the range $[0, m(\Omega)]$. Indeed, as shown in [2], $m(\Omega)$ is the largest value for α such that the de-discounting operation (12) yields a BBA. De-discounting m with this maximal value is called *maximal de-discounting*. The result is the *totally reinforced belief function*, noted ${}^{tr}m$ and defined as follows:

$${}^{tr}m(A) = \begin{cases} \frac{m(A)}{1-m(\Omega)} & \forall A \subset \Omega, \\ 0 & \text{otherwise.} \end{cases} \tag{13}$$

The mass function ${}^{tr}m$ is thus obtained from m by redistributing the mass $m(\Omega)$ among the strict subsets of Ω.

3.3 Extended Discounting Scheme

In [20], Zhu and Basir proposed to extend the discounting process in order to *strengthen, discount or contradict* belief functions. The extended discounting scheme is composed of two transformations.

The first transformation, allowing us to strengthen or weaken a source of information, is introduced by retaining the discounting equation (9), while allowing the discount rate α to be in the range $\left[\frac{-m(\Omega)}{1-m(\Omega)}, 1\right]$.

- If $\alpha \in [0,1]$, this transformation is the discounting operation.
- If $\alpha \in [\frac{-m(\Omega)}{1-m(\Omega)}, 0]$, this transformation is equivalent to the de-discounting operation (12) with the reparameterization $\alpha = \frac{-\alpha'}{1-\alpha'}$ with $\alpha' \in [0, m(\Omega)]$. Indeed,

$${}^\alpha m = \left(1 - \frac{-\alpha'}{1-\alpha'}\right) m + \frac{-\alpha'}{1-\alpha'} m_\Omega = \frac{m - \alpha' m_\Omega}{1-\alpha'}\ . \tag{14}$$

The second transformation, allowing us to contradict a non-vacuous and normal belief function m, is defined by the following equation:

$$\begin{cases} {}^{\alpha}m(\overline{A}) &= (\alpha-1)m(A) \quad \text{if } A \subset \Omega, \\ {}^{\alpha}m(\Omega) &= (\alpha-1)m(\Omega)+2-\alpha \text{ otherwise,} \end{cases} \tag{15}$$

where $\alpha \in \left[1, 1+\frac{1}{1-m(\Omega)}\right]$.

- If $\alpha = 1$, ${}^{\alpha}m = m_{\Omega}$.
- If $\alpha = 1+\frac{1}{1-m(\Omega)}$, ${}^{\alpha}m = \overline{{}^{tr}m}$, where $\overline{m}$ denotes the negation of m [4], defined by $\overline{m}(A) = m(\overline{A})$, $\forall A \subseteq \Omega$. In other words, after being totally reinforced, each basic belief mass $m(A)$ is transferred to its complement. The BBA m is then fully contradicted.

This scheme has been successfully applied in medical imaging [20]. However, it suffers from a lack of formal justification. Indeed, the number $(1-\alpha)$ can no longer be interpreted as a degree of belief as it can take values greater than 1 and smaller than 0.

3.4 Contextual Discounting based on a coarsening

Contextual discounting was introduced in [10]. It makes it possible to take into account the fact that the reliability of the source of information can be expected to depend on the true answer of the question of interest.

For instance, in medical diagnosis, depending on his/her specialty, experience or training, a physician may be more or less competent to diagnose some types of diseases. Likewise, in target recognition, a sensor may be more capable of recognizing some types of targets while being less effective for other types.

Let $\Theta = \{\theta_1, \ldots, \theta_L\}$ be a coarsening [13, chapter 6] of Ω, in other words $\theta_1, \ldots, \theta_L$ form a partition of Ω.

Unlike (10), in the contextual model, agent Ag is assumed to hold beliefs on the reliability of the source of information conditionally on each θ_ℓ, $\ell \in \{1, \ldots, L\}$:

$$\begin{cases} m_{Ag}^{\mathscr{R}}[\theta_\ell](\{R\}) = 1-\alpha_\ell = \beta_\ell \\ m_{Ag}^{\mathscr{R}}[\theta_\ell](\mathscr{R}) \quad = \alpha_\ell \ . \end{cases} \tag{16}$$

For all $\ell \in \{1, \ldots, L\}$, $\beta_\ell + \alpha_\ell = 1$, and β_ℓ represents the degree of belief that the source is reliable knowing that the true answer of the question of interest belongs to θ_ℓ.

In the same way as in the discounting operation (9), agent Ag considers that the source can be in two states: reliable or not reliable [14, 10]:

- If the source is reliable (state R), the information m_S^{Ω} it provides becomes Ag's knowledge. Formally, $m_{Ag}^{\Omega}[\{R\}] = m_S^{\Omega}$.

- If the source is not reliable (state NR), the information m_S^Ω it provides is discarded, and Ag remains in a state of ignorance: $m_{Ag}^\Omega[\{NR\}] = m_\Omega$.

The knowledge held by agent Ag, based on the information m_S^Ω from a source S as well as metaknowledge $m_{Ag}^{\mathscr{R}}$ concerning the reliability of the source can then be computed by:

- Deconditioning the L BBAs $m_{Ag}^{\mathscr{R}}[\theta_\ell]$ on the product space $\Omega \times \mathscr{R}$ using (8);
- Deconditioning $m_{Ag}^\Omega[\{R\}]$ on the same product space $\Omega \times \mathscr{R}$ using (8) as well;
- Combining them using the CRC (2);
- Marginalizing the result on Ω using (4).

Formally:

$$m_{Ag}^\Omega[m_S^\Omega, m_{Ag}^{\mathscr{R}}] = \left(\bigcirc\!\!\!\!\cap_{\ell=1}^{L} m_{Ag}^{\mathscr{R}}[\theta_\ell]^{\Uparrow \Omega\times\mathscr{R}} \bigcirc\!\!\!\!\cap\, m_{Ag}^\Omega[\{R\}]^{\Uparrow \Omega\times\mathscr{R}} \right)^{\downarrow\Omega} . \tag{17}$$

As shown in [10], the resulting BBA m_{Ag}^Ω , only depends on m_S and on the vector $\alpha = (\alpha_1, \ldots, \alpha_L)$ of discount rates. It is then denoted by ${}^{\alpha}_{\Theta}m$.

Proposition 1 ([10, Proposition 8]). *The contextual discounting ${}^{\alpha}_{\Theta}m$ on a coarsening Θ of a BBA m is equal to the disjunctive combination of m with a BBA m_0^Ω such that:*

$$m_0^\Omega = m_1^\Omega \bigcirc\!\!\!\!\cup\, m_2^\Omega \bigcirc\!\!\!\!\cup \ldots \bigcirc\!\!\!\!\cup\, m_L^\Omega \ , \tag{18}$$

where each m_ℓ^Ω, $\ell \in \{1, \ldots, L\}$, is defined by:

$$m_\ell^\Omega(A) = \begin{cases} \beta_\ell & \text{if } A = \emptyset \\ \alpha_\ell & \text{if } A = \theta_\ell \\ 0 & \text{otherwise.} \end{cases} \tag{19}$$

Remark 1. Two special cases of this discounting operation can be considered.

- If $\Theta = \{\Omega\}$ denotes the trivial partition of Ω in one class, combining m with m_0 defined by (11) is equivalent to combining m with m_0 defined by (18), so this contextual discounting operation is identical to the classical discounting operation.
- If $\Theta = \Omega$, the finest partition of Ω, this discounting is simply called contextual discounting and denoted ${}^{\alpha}m$. It is defined by the disjunctive combination of m with the BBA $m_1^\Omega \bigcirc\!\!\!\!\cup\, m_2^\Omega \bigcirc\!\!\!\!\cup \ldots \bigcirc\!\!\!\!\cup\, m_K^\Omega$, where each m_k^Ω, $k \in \{1, \ldots, K\}$ is defined by $m_k^\Omega(\emptyset) = \beta_k$ and $m_k^\Omega(\{\omega_k\}) = \alpha_k$.

In the following section, a new parameterized family of transformations encompassing all the non-contextual schemes presented in this section, is introduced and justified.

4 A parameterized family of correction mechanisms

In this section, the hypotheses concerning the states in which agent Ag considers that the source can be, are extended in the following way.

Let us assume that the source can be in N states R_i, $i \in \{1,\ldots,N\}$, whose interpretations are given by transformations m_i of m: if the source is in the state R_i then $m^{\Omega}_{Ag} = m_i$.

$$m^{\Omega}_{Ag}[\{R_i\}] = m_i, \quad \forall i \in \{1,\ldots,N\} \ . \tag{20}$$

Let $\mathscr{R} = \{R_1,\ldots,R_N\}$, and let us suppose that, for all $i \in \{1,\ldots,N\}$:

$$m^{\mathscr{R}}_{Ag}(\{R_i\}) = \nu_i, \text{ with } \sum_{i=1}^{N} \nu_i = 1 \ . \tag{21}$$

The knowledge held by agent Ag, based on the information m^{Ω}_S from a source S and on metaknowledge $m^{\mathscr{R}}_{Ag}$ regarding the different states in which the source can be, can then be computed by:

- Deconditioning the N BBAs $m^{\Omega}_{Ag}[\{R_i\}]$ on the product space $\Omega \times \mathscr{R}$ using (8);
- Vacuously extending $m^{\mathscr{R}}_{Ag}$ on the same product space $\Omega \times \mathscr{R}$ using (5);
- Combining all BBAs using the CRC (2);
- Marginalizing the result on Ω using (4).

Formally:

$$m^{\Omega}_{Ag}[m^{\Omega}_S, m^{\mathscr{R}}_{Ag}] = \left(\textcircled{\cap}_{i=1}^{N} m^{\Omega}_{Ag}[\{R_i\}]^{\Uparrow \Omega\times\mathscr{R}} \textcircled{\cap} m^{\mathscr{R}\uparrow\Omega\times\mathscr{R}}_{Ag} \right)^{\downarrow\Omega} \ . \tag{22}$$

Proposition 2. *The BBA m^{Ω}_{Ag} defined by (22) only depends on m_i and ν_i, $i \in \{1,\ldots,N\}$. The result is noted ${}^{\nu}m$, ν denoting the vector of ν_i, and verifies:*

$$m^{\Omega}_{Ag} = {}^{\nu}m = \sum_{i=1}^{N} \nu_i \, m_i \ . \tag{23}$$

Proof. For all $i \in \{1,\ldots,N\}$ and $A \subseteq \Omega$:

- from (5) and (21), the vacuous extension of $m^{\mathscr{R}}_{Ag}$ is given by:

$$m^{\mathscr{R}\uparrow\Omega\times\mathscr{R}}_{Ag}(\Omega \times \{R_i\}) = \nu_i \ ; \tag{24}$$

- from (8) and (20), the deconditioning of $m^{\Omega}_{Ag}[\{R_i\}]$ verifies:

$$m^{\Omega}_{Ag}[\{R_i\}]^{\Uparrow\Omega\times\mathscr{R}}(A \times \{R_i\} \cup \Omega \times \overline{\{R_i\}}) = m_i(A) \ . \tag{25}$$

However, $\forall i \in \{1,\ldots,N\}$ and $\forall A_i \subseteq \Omega$:

$$\cap_{i=1}^{N}(A_i \times \{R_i\} \cup \Omega \times \overline{\{R_i\}}) = \cup_{i=1}^{N} A_i \times \{R_i\} \ , \tag{26}$$

and, $\forall j \in \{1,\ldots,N\}$:

$$(\cup_{i=1}^{N} A_i \times \{R_i\}) \cap \Omega \times \{R_j\} = A_j \times \{R_j\} \ . \tag{27}$$

Therefore, the conjunctive combination of $m_{Ag}^{\Omega}[\{R_i\}]^{\Uparrow \Omega \times \mathscr{R}}$, $i \in \{1,\ldots,N\}$, with $m_{Ag}^{\mathscr{R} \uparrow \Omega \times \mathscr{R}}$, denoted $\circledcap m_{Ag}^{\Omega \times \mathscr{R}}$, has N focal elements such that:

$$\circledcap m_{Ag}^{\Omega \times \mathscr{R}}(A_j \times \{R_j\}) = \nu_j \ m_j(A_j) \prod_{i \neq j} \underbrace{\sum_{A \subseteq \Omega} m_i(A)}_{=1}, \quad \forall j \in \{1,\ldots,N\} \ , \tag{28}$$

or, equivalently, $\forall A \subseteq \Omega$ and $\forall i \in \{1,\ldots,N\}$:

$$\circledcap m_{Ag}^{\Omega \times \mathscr{R}}(A \times \{R_i\}) = \nu_i \ m_i(A). \tag{29}$$

Then, after projecting onto Ω:

$$m_{Ag}^{\Omega}(A) = \sum_{i=1}^{N} \nu_i \ m_i(A) \quad \forall A \subseteq \Omega, \tag{30}$$

which completes the proof. □

Proposition 3. *Discounting, de-discounting and extended discounting operations are particular cases of correction mechanisms expressed by (23):*

- *Discounting corresponds to the case of two states R_1 and R_2 such that $m_1 = m_\Omega$ and $m_2 = m$ (as already exposed in Section 3.1).*
- *De-discounting corresponds to the case of two states such that $m_1 = m$ and $m_2 = {}^{tr}m$, which means a first state where the information provided by the source is accepted, and a second one where this information is totally reinforced.*
- *The first transformation of the extended discounting operation, discounting equation (9) with $\alpha \in [-m(\Omega)/(1-m(\Omega)), 1]$, is obtained in the particular case of two states such that $m_1 = m_\Omega$ and $m_2 = {}^{tr}m$.*
- *The second transformation of the extended discounting operation (15) is retrieved by considering two states: a first one where the source is fully contradicted ($m_1 = \overline{{}^{tr}m}$) [18], and a second one where the information provided by the source is rejected ($m_2 = m_\Omega$).*

Proof. By considering two states such that $m_1 = m$ and $m_2 = {}^{tr}m$, ${}^{\nu}m = \nu_1 m + \nu_2 {}^{tr}m$ is a reparameterization of the de-discounting operation (12) with $\nu_1 = \frac{m(\Omega)-\alpha}{(1-\alpha)m(\Omega)}$, $\alpha \in [0, m(\Omega)]$. Indeed:

$$
\begin{aligned}
{}^{v}m(A) &= \frac{m(\Omega)-\alpha}{(1-\alpha)m(\Omega)}\, m(A) + \left(1-\frac{m(\Omega)-\alpha}{(1-\alpha)m(\Omega)}\right)\frac{m(A)}{1-m(\Omega)} \\
&= \frac{m(\Omega)-\alpha}{(1-\alpha)m(\Omega)}\, m(A) + \frac{(1-\alpha)m(\Omega)-m(\Omega)+\alpha}{(1-\alpha)m(\Omega)}\frac{m(A)}{1-m(\Omega)} \\
&= \frac{m(\Omega)-\alpha}{(1-\alpha)m(\Omega)}\, m(A) + \frac{\alpha\,(1-m(\Omega))}{(1-\alpha)m(\Omega)}\frac{m(A)}{1-m(\Omega)} \\
&= \frac{m(A)}{1-\alpha} \quad \forall A \subset \Omega,
\end{aligned}
$$

and

$$
{}^{v}m(\Omega) = \frac{m(\Omega)-\alpha}{(1-\alpha)m(\Omega)}\, m(\Omega) = \frac{m(\Omega)-\alpha}{1-\alpha}.
$$

The first transformation of the extended discounting operation, Equation (9) with $\alpha \in [-m(\Omega)/(1-m(\Omega)), 1]$, concerning the discounting or reinforcement of the source, is a reparameterization of (23) in the particular case of two states such that $m_1 = m_\Omega$ and $m_2 = {}^{tr}m$ with $\nu_1 = (1-\alpha)m(\Omega)+\alpha$. Indeed:

$$
\begin{aligned}
{}^{v}m(A) &= (1-(1-\alpha)m(\Omega)-\alpha)\frac{m(A)}{1-m(\Omega)} = (1-\alpha)\, m(A), \quad \forall A \subset \Omega, \\
{}^{v}m(\Omega) &= (1-\alpha)m(\Omega)+\alpha.
\end{aligned}
$$

Finally, the second transformation of the extended discounting operation (15), allowing one to contradict a source, is also a reparameterization of (23) by considering two states such that $m_1 = \overline{{}^{tr}m}$ and $m_2 = m_\Omega$, and setting $\nu_1 = (\alpha-1)(1-m(\Omega))$ with $\alpha \in [1, 1+\frac{1}{1-m(\Omega)}]$:

$$
\begin{aligned}
{}^{v}m(\overline{A}) &= (\alpha-1)(1-m(\Omega))\frac{m(A)}{1-m(\Omega)} = (\alpha-1)\, m(A), \quad \forall A \subset \Omega, \\
{}^{v}m(\Omega) &= 1-(\alpha-1)(1-m(\Omega)) = 1-\alpha+\alpha\, m(\Omega)+1-m(\Omega) \\
&= (\alpha-1)m(\Omega)+2-\alpha.
\end{aligned}
$$

□

Models of correction mechanisms corresponding to discounting, de-discounting and extended discount operations are summarized in Table 1.

Table 1 Models corresponding to the correction mechanisms presented in Section 3.

Interpretations		Operation
$m_1 = m_\Omega$	$m_2 = m$	discounting
$m_1 = m$	$m_2 = {}^{tr}m$	de-discounting
$m_1 = m_\Omega$	$m_2 = {}^{tr}m$	extended disc. (1)
$m_1 = \overline{{}^{tr}m}$	$m_2 = m_\Omega$	extended disc. (2)

Remark 2. The first transformation of the extended discounting operation is a discounting of ${}^{tr}m$, while the second transformation is a discounting of $\overline{{}^{tr}m}$.

Remark 3. De-discounting operation is a particular reinforcement process. A more informative reinforcement than ${}^{tr}m$ can be chosen, for instance, the "pignistic BBA" defined, $\forall \omega \in \Omega$, by:

$$
{}^{bet}m(\{\omega\}) = \sum_{\{A \subseteq \Omega, \omega \in A\}} \frac{m(A)}{(1-m(\emptyset))|A|} \ . \tag{31}
$$

Thus, another reinforcement process is given by:

$$
{}^{v}m = v_1 m + (1-v_1)\,{}^{bet}m \ . \tag{32}
$$

Remark 4. By choosing $m^{\mathscr{R}}_{Ag}$ as follows:

$$
\begin{cases} m^{\mathscr{R}}_{Ag}(\{R_i\}) = v_i & \forall i \in \{1,\ldots,N\}, \\ m^{\mathscr{R}}_{Ag}(\mathscr{R}) \quad = 1 - \sum_{i=1}^{N} v_i, & \end{cases} \tag{33}
$$

with $\sum_{i=1}^{N} v_i \leq 1$, Equation (22) leads to:

$$
{}^{v}m = \sum_{i=1}^{N} v_i \, m_i + (1 - \sum_{i=1}^{N} v_i) m_\Omega \ , \tag{34}
$$

which is similar to (23) if one considers a state such that $m_i = m_\Omega$.

5 Correction mechanisms based on decompositions

The preceding section has introduced a general form of correction mechanisms encompassing, in particular, the discounting, de-discounting and extended discounting operations. As mentioned in Remark 1 in Section 3.4, the discounting operation can also be seen as a particular case of the contextual discounting. However, the contextual discounting does not belong to the family of correction mechanisms presented in the previous section. In this section, contextual discounting is shown to be a particular member of another family of correction mechanisms based on the disjunctive decomposition of a subnormal BBA introduced by Denœux in [3].

5.1 Canonical conjunctive and disjunctive decompositions

In [16], extending the notion of separable BBA introduced by Shafer [13, chapter 4], Smets shows that each non-dogmatic BBA m can be uniquely decomposed into a conjunctive combination of *generalized simple BBAs (GSBBAs)*, denoted $A^{w(A)}$ with

$A \subset \Omega$, and defined from 2^Ω to $\mathbb{R}$ by:

$$A^{w(A)} : \begin{array}{l} \Omega \mapsto w(A) \\ A \mapsto 1 - w(A) \\ B \mapsto 0 \ , \ \forall B \in 2^\Omega \setminus \{A, \Omega\} \ , \end{array} \tag{35}$$

with $w(A) \in [0, \infty)$.

Every non-dogmatic BBA m can then be canonically decomposed into a conjunctive combination of GSBBAs:

$$m = \bigcirc\!\!\!\!\cap_{A \subset \Omega} A^{w(A)} \ . \tag{36}$$

In [3], Denœux introduces another decomposition: the canonical disjunctive decomposition of a subnormal BBA into *negative GSBBAs (NGSBBAs)*, denoted $A_{v(A)}$ with $A \supset \emptyset$, and defined from 2^Ω to $\mathbb{R}$ by:

$$A_{v(A)} : \begin{array}{l} \emptyset \mapsto v(A) \\ A \mapsto 1 - v(A) \\ B \mapsto 0 \ , \ \forall B \in 2^\Omega \setminus \{\emptyset, A\} \ , \end{array} \tag{37}$$

with $v(A) \in [0, \infty)$.

Every subnormal BBA m can be canonically decomposed into a disjunctive combination of NGSBBAs:

$$m = \bigcirc\!\!\!\!\cup_{A \supset \emptyset} A_{v(A)} \ . \tag{38}$$

Indeed, as remarked in [3], the negation of m can also be conjunctively decomposed as soon as m is subnormal (in this case, $\overline{m}$ is non-dogmatic). Then:

$$\overline{m} = \bigcirc\!\!\!\!\cap_{A \subset \Omega} A^{\overline{w}(A)} \Rightarrow m = \overline{\bigcirc\!\!\!\!\cap_{A \subset \Omega} A^{\overline{w}(A)}} = \bigcirc\!\!\!\!\cup_{A \subset \Omega} \overline{A^{\overline{w}(A)}} = \bigcirc\!\!\!\!\cup_{A \supset \emptyset} A_{\overline{w}(\overline{A})} \ . \tag{39}$$

The relation between functions v and w is then $v(A) = \overline{w}(\overline{A})$ for all $A \supset \emptyset$.

5.2 A correction mechanism based on the disjunctive decomposition

According to the previous definitions (35) and (37), BBAs m_ℓ, $\ell \in \{1, \ldots, L\}$, defined in (19) by $m_\ell(\emptyset) = \beta_\ell$ and $m_\ell(\theta_\ell) = \alpha_\ell$, can be denoted $\theta_{\ell_{\beta_\ell}}$ or θ_{β_ℓ} in a simple way.

From (18) and (38), the contextual discounting on a coarsening $\Theta = \{\theta_1, \ldots, \theta_L\}$ of Ω of a subnormal BBA m is thus defined by:

$${}^{\alpha}_{\Theta}m = m \bigcirc\!\!\!\!\cup \theta_{\beta_1} \bigcirc\!\!\!\!\cup \ldots \bigcirc\!\!\!\!\cup \theta_{\beta_L} = \bigcirc\!\!\!\!\cup_{A \supset \emptyset} A_{v(A)} \bigcirc\!\!\!\!\cup \theta_{\beta_1} \bigcirc\!\!\!\!\cup \ldots \bigcirc\!\!\!\!\cup \theta_{\beta_L} .$$

In particular, as $A_{v_1(A)} \bigcirc\!\!\!\!\cup A_{v_2(A)} = A_{v_1 v_2(A)}$ for all non empty subet A of Ω:

- The classical discounting of a subnormal BBA $m = \bigcirc\!\!\!\!\cup_{A \supset \emptyset} A_{v(A)}$ is defined by:

$$^{\alpha}m = \Omega_{\beta v(\Omega)} \bigcirc\!\!\!\!\cup_{\Omega \supset A \supset \emptyset} A_{v(A)} \ ; \tag{40}$$

- The contextual discounting (Remark 1) of a subnormal BBA $m = \bigcirc\!\!\!\!\cup_{A \supset \emptyset} A_{v(A)}$ is defined by:

$$^{\alpha}m = \bigcirc\!\!\!\!\cup_{\omega_k \in \Omega} \{\omega_k\}_{\beta_k v(\{\omega_k\})} \bigcirc\!\!\!\!\cup_{A \subset \Omega, |A|>1} A_{v(A)} \ . \tag{41}$$

These contextual discounting operations are then particular cases of a more general correction mechanism defined by:

$$^{\alpha\cup}m = \bigcirc\!\!\!\!\cup_{A \supset \emptyset} A_{\beta_A v(A)}, \tag{42}$$

where $\beta_A \in [0,1]$ for all $A \supset \emptyset$ and α is the vector $\{\alpha_A\}_{A \supset \emptyset}$.

In [10], the interpretation of each β_A has been given only in the case where the union of the subsets A forms a partition of Ω, β_A being interpreted as the degree of belief held by the agent regarding the fact that the source is reliable, knowing that the value searched belongs to A.

Instead of considering (16), let us now suppose that agent Ag holds beliefs regarding the reliability of the source, conditionally on each subset A of Ω:

$$\begin{cases} m_{Ag}^{\mathscr{R}}[A](\{R\}) = 1 - \alpha_A = \beta_A \\ m_{Ag}^{\mathscr{R}}[A](\mathscr{R}) \ \ = \alpha_A \ , \end{cases} \tag{43}$$

where $\alpha_A \in [0,1]$.

In the same way as in Section 3.4, the knowledge held by agent Ag, based on the information m_S^{Ω} from a source and on metaknowledge $m_{Ag}^{\mathscr{R}}$ (43) regarding the reliability of this source, can be computed as follows:

$$m_{Ag}^{\Omega}[m_S^{\Omega}, m_{Ag}^{\mathscr{R}}] = \left(\bigcirc\!\!\!\!\cap_{A \subseteq \Omega} m_{Ag}^{\mathscr{R}}[A]^{\Uparrow \Omega \times \mathscr{R}} \bigcirc\!\!\!\!\cap\, m_{Ag}^{\Omega}[\{R\}]^{\Uparrow \Omega \times \mathscr{R}} \right)^{\downarrow \Omega} . \tag{44}$$

Proposition 4. *The BBA m_{Ag}^{Ω} resulting from (44) only depends on m_S^{Ω} and the vector $\alpha = \{\alpha_A\}_{A \subseteq \Omega}$. The result is denoted ${}^{\alpha}_{2^\Omega}m$ and is equal to the disjunctive combination of m_S^{Ω} with a BBA m_0^{Ω} defined by:*

$$m_0^{\Omega}(C) = \prod_{\cup A = C} \alpha_A \prod_{\cup B = \overline{C}} \beta_B, \quad \forall C \subseteq \Omega. \tag{45}$$

Proof. For each $A \subseteq \Omega$, the deconditioning of $m_{Ag}^{\mathscr{R}}[A]$ on $\Omega \times \mathscr{R}$ is given by:

$$m_{Ag}^{\mathscr{R}}[A]^{\Uparrow \Omega \times \mathscr{R}}(A \times \{R\} \cup \overline{A} \times \mathscr{R}) = \beta_A, \tag{46}$$

$$m_{Ag}^{\mathscr{R}}[A]^{\Uparrow \Omega \times \mathscr{R}}(\Omega \times \mathscr{R}) = \alpha_A. \tag{47}$$

With $A \neq B$:

$$(A \times \{R\} \cup \overline{A} \times \mathscr{R}) \cap (B \times \{R\} \cup \overline{B} \times \mathscr{R}) = (A \cup B) \times \{R\} \cup \overline{(A \cup B)} \times \mathscr{R} \ .$$

Then:

$$\textcircled{\cap}_{A\subseteq\Omega} m_{Ag}^{\mathscr{R}}[A]^{\Uparrow\Omega\times\mathscr{R}}(C\times\{R\}\cup\overline{C}\times\mathscr{R}) = \prod_{\cup D=\overline{C}}\alpha_D \prod_{\cup E=C}\beta_E, \quad \forall C\subseteq\Omega\ , \tag{48}$$

or, by exchanging the roles of C and $\overline{C}$:

$$\textcircled{\cap}_{A\subseteq\Omega} m_{Ag}^{\mathscr{R}}[A]^{\Uparrow\Omega\times\mathscr{R}}(\overline{C}\times\{R\}\cup C\times\mathscr{R}) = \prod_{\cup D=C}\alpha_D \prod_{\cup E=\overline{C}}\beta_E, \quad \forall C\subseteq\Omega\ . \tag{49}$$

It remains to combine conjunctively $m_{Ag}^{\Omega}[\{R\}]^{\Uparrow\Omega\times\mathscr{R}}$ and $\textcircled{\cap}_{A\subseteq\Omega} m_{Ag}^{\mathscr{R}}[A]^{\Uparrow\Omega\times\mathscr{R}}$ which have focal sets of the form $B\times\{R\}\cup\Omega\times\{NR\}$ and $\overline{C}\times\{R\}\cup C\times\mathscr{R}$, respectively, with $B,C\subseteq\Omega$. The intersection of two such focal sets is:

$$(\overline{C}\times\{R\}\cup C\times\mathscr{R})\cap(B\times\{R\}\cup\Omega\times\{NR\}) = B\times\{R\}\cup C\times\{NR\}\ ,$$

and it can be obtained only for a particular choice of B and C. Then:

$$\textcircled{\cap}_{A\subseteq\Omega} m_{Ag}^{\mathscr{R}}[A]^{\Uparrow\Omega\times\mathscr{R}} \textcircled{\cap} m_{Ag}^{\Omega}[\{R\}]^{\Uparrow\Omega\times\mathscr{R}}(B\times\{R\}\cup C\times\{NR\}) = \left[\prod_{\cup D=C}\alpha_D \prod_{\cup E=\overline{C}}\beta_E\right] m_S^{\Omega}(B).$$

Finally, the marginalization of this BBA on Ω is given by:

$$^{\alpha}m(A) = \sum_{B\cup C=A}\left[\prod_{\cup D=C}\alpha_D \prod_{\cup E=\overline{C}}\beta_E\right] m_S^{\Omega}(B), \quad \forall A\subseteq\Omega, \tag{50}$$

□

Let us note that the above proof has many similarities with proofs presented in [10, Sections A.1 and A.3].

As in the case of contextual discounting operations considered in Section 3.4, the BBA m_0^{Ω} defined in Proposition 4 admits a simple decomposition described in the following proposition.

Proposition 5. *The BBA m_0^{Ω} defined in Proposition 4 can be rewritten as:*

$$m_0^{\Omega} = \textcircled{\cup}_{A\supset\emptyset} A_{\beta_A}\ . \tag{51}$$

Proof. Directly from (45) and the definition of the DRC (3). □

From (51), the contextual discounting ${}^{\alpha}_{2^\Omega}m$ of a subnormal BBA $m = \textcircled{\cup}_{A\supset\emptyset} A_{\nu(A)}$ is defined by:

$${}^{\alpha}_{2^\Omega}m = \textcircled{\cup}_{A\supset\emptyset} A_{\nu(A)} \textcircled{\cup}_{A\supset\emptyset} A_{\beta_A} = \textcircled{\cup}_{A\supset\emptyset} A_{\beta_A\nu(A)} = {}^{\alpha\cup}m\ . \tag{52}$$

This contextual discounting is thus equivalent to the correction mechanism introduced in this section. Each coefficient β_A of this correction mechanism can then be interpreted as the degree of belief held by the agent Ag regarding the fact that the source is reliable knowing that the true answer to the question Q of interest belongs to A.

5.3 A correction mechanism based on the conjunctive decomposition

In a similar way, a correction mechanism for a non-dogmatic BBA m can be defined, from the conjunctive decomposition of m, by:

$$^{\alpha\cap}m = \bigcirc\!\!\!\!\cap_{A\subset\Omega} A^{\beta_A w(A)} \;; \tag{53}$$

where $\forall A \subset \Omega, \beta_A \in [0,1]$, and α is the vector $\{\alpha_A\}_{A\subset\Omega}$.

Correction mechanisms $^{\alpha\cap}m$ (42) and $^{\alpha\cup}m$ (53) are related in the following way. Let us consider a subnormal BBA m, $\overline{m}$ is then non-dogmatic:

$$^{\alpha\cap}\overline{m} = \bigcirc\!\!\!\!\cap_{A\subset\Omega} A^{\beta_A \overline{w}(A)} \;. \tag{54}$$

Then:

$$\begin{aligned} \overline{^{\alpha\cap}\overline{m}} &= \overline{\bigcirc\!\!\!\!\cap_{A\subset\Omega} A^{\beta_A \overline{w}(A)}} \\ &= \bigcirc\!\!\!\!\cup_{A\subset\Omega} \overline{A^{\beta_A \overline{w}(A)}} \\ &= \bigcirc\!\!\!\!\cup_{A\supset\emptyset} A_{\beta_A \overline{w}(\overline{A})} \\ &= \bigcirc\!\!\!\!\cup_{A\supset\emptyset} A_{\beta_A v(A)} \\ &= {}^{\alpha\cup}m \end{aligned} \tag{55}$$

These two correction mechanisms can thus be seen as belonging to a general family of correction mechanisms.

6 Application example

In this section, an application example in the domain of postal address recognition illustrates the potential benefits of using a particular correction mechanism of the form (23).

In this application, three postal address readers (PARs) are available, each one providing pieces of information regarding the address lying on the image of a mail. These pieces of knowledge are represented by belief functions on a frame of discernment gathering all postal addresses. Belief functions can then be combined in order to make a decision. This fusion scheme is represented in Fig. 1. Details of this application can be found in [11].

An extension of this scheme is obtained by considering the fact that PAR 1 and PAR 2 each output an address and a confidence score regarding the town part of the address.

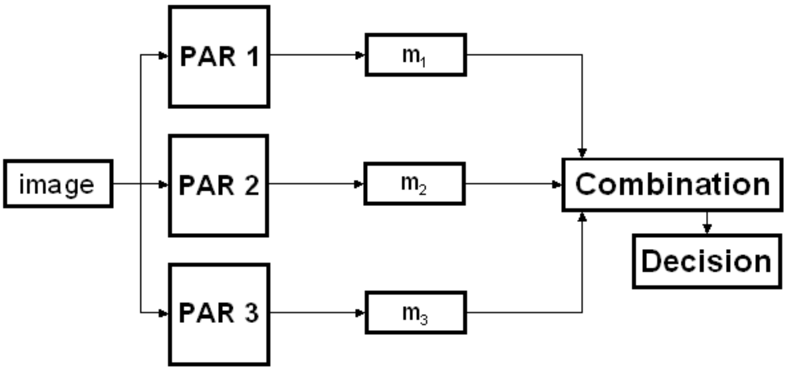

Fig. 1 Fusion scheme with three PARs in the belief function framework.

To visualize the real information provided by these confidence scores, scores of correct and incorrect towns output by PAR 1 for a set of postal addresses are shown in Fig. 2.

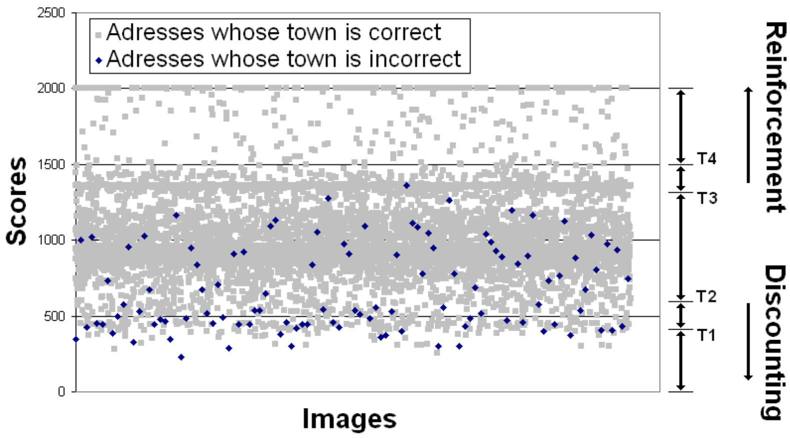

Fig. 2 Confidence scores and addresses provided by PAR 1 regarding images of a learning set. A dark rhomb corresponds to an address whose town is incorrect. A clear square is associated with an address whose town is correct.

It can be observed that the greater the score is, the more important is the proportion of addresses with a correct town. Hence, this score carries useful information regarding the reliability of the town information in the output address. Similar observations were made with PAR 2. Therefore, BBAs m_1 and m_2 representing the information provided by PAR 1 and PAR 2, should be corrected according to these scores. An idea consists in reinforcing the information provided by a PAR when the score is high, and, conversely, discounting it when the score is too low. For that purpose, we defined four thresholds T_1, T_2, T_3, and T_4 illustrated in Fig. 2, such that information provided by the PAR is:

- totally discounted, if the score is lower than T_1;
- discounted according to the score, if the score belongs to $[T_1, T_2]$;
- kept unchanged, if the score belongs to $[T_2, T_3]$;
- reinforced according to the score, if the score belongs to $[T_3, T_4]$;
- at last, totally reinforced, if the score is greater than T_4.

Formally, this adjustment can be realized, for both PARs 1 and 2 (Fig. 3), by using the correction mechanism defined by:

$$^{v}m = v_1\, m_{\Omega} + v_2\, m + v_3\, {}^{tr}m \;, \tag{56}$$

where parameters v_i are set as illustrated in Fig. 4.

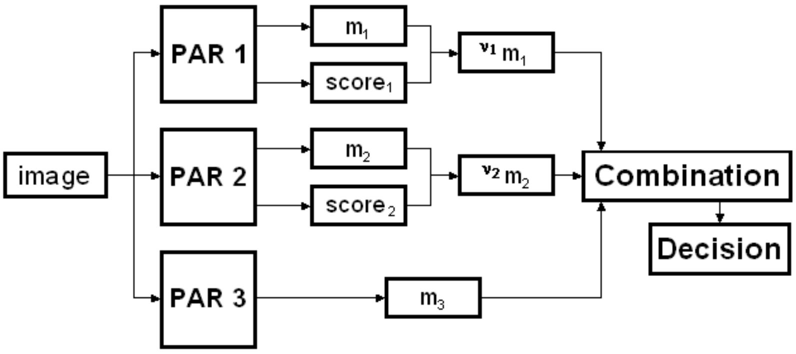

Fig. 3 An extended model adjusting BBAs provided by *PAR* 1 and *PAR* 2 according to supplied scores.

Performances of this combination, on a test set of mails, are reported in Fig. 5. To preserve the confidentiality of PARs performances, reference values were used when representing performance rates. Correct recognition rates, represented on the x-axis, are expressed relatively to a reference correct recognition rate, denoted by R. Error rates, represented on the y-axis, are expressed relatively to a reference error rate, denoted by E. The rate R has a value greater than 80%. The rate E has a value smaller than 0.1%.

As different PARs are available in this application, we can expect the combination to yield the greatest possible recognition rates while keeping error rate at an acceptable level. In this article, the maximal tolerated error rate is chosen equal to the least PARs error rates.

This extended model allows us to obtain a combination point denoted by C_+, which is associated with an acceptable error rate and a higher recognition rate than

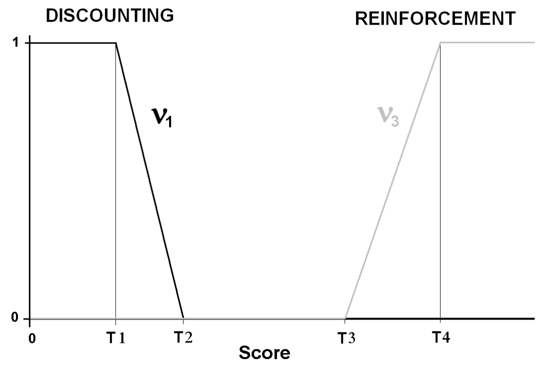

Fig. 4 Correction parameters as function of the scores ($\nu_1 + \nu_2 + \nu_3 = 1$).

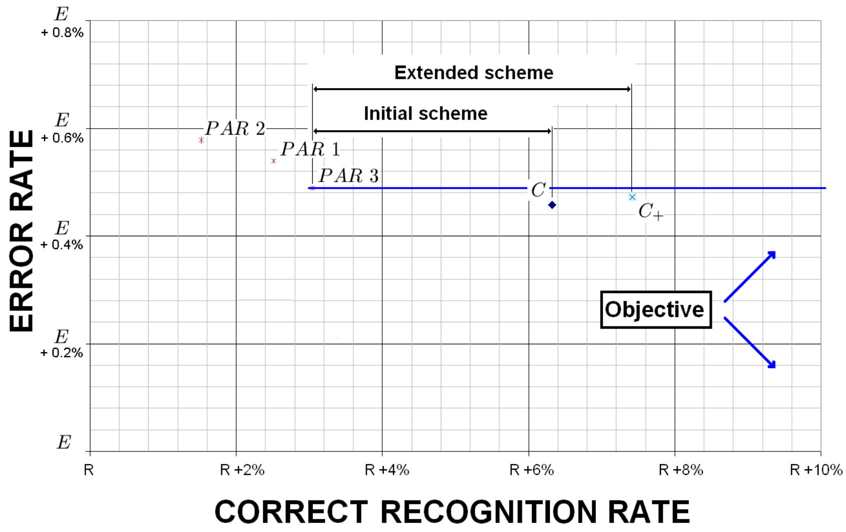

Fig. 5 PARs and combination performances regarding towns written on mails.

the previous combination point C, obtained with the model illustrated in Fig. 1. The individual performances of PARs are further improved using the extended model based on a correction mechanism.

In [8], another example of improvement in the same application can be found using contextual discounting.

7 Conclusion

In this article, two families of belief function correction mechanisms have been introduced and justified.

The first family of correction mechanisms highlights the links between the discounting, the de-discounting, and the extended discounting, and generalizes these three operations. Different transformations, expressed by belief functions, can be associated to different states in which the source can be: reliable, not reliable, too cautious, lying, etc.

The second family, based on the concepts of negation of a BBA and disjunctive and conjunctive decompositions of a BBA, generalizes the contextual discounting operation.

An application example has illustrated a practical interest of the first family. It introduces a way to combine scores with decisions to improve the recognition performances.

Future works will aim at exploring more deeply the second family of correction mechanisms and testing it on real data.

It would also be interesting to automatically learn the coefficients of the correction mechanisms from data, as done in the "expert tuning" method for the classical or the contextual discounting operations [5, 10].

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